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Modal Analysis of Controlled Multilink Systems with Flexible Links and Joints

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In this study, a systematic finite element transfer matrix scheme for the modal analysis of integrated multilink control systems with flexible links and joints is presented. This proposed scheme treats the control signals as parts of physical degrees of freedom of the system and then modifies the transfer matrices considering the coupling and noncoupling characteristics between the electric and mechanical interfaces. Since it is systematic, the effects of interaction between control and structure can be considered, and reference control signals can be directly input to the system without calculating equivalent static actuating torques. An example of a two-link/two-joint control system is given to illustrate its availability in sensitivity analysis of both structural and control design parameters by a systematic manner. The illustrated example also shows its capability in the application of both collocated and noncollocated control simulations.

I. Introduction

ACCURATE response prediction and effective control of flexible mechanical structures are important in control engineering, especially in the field of space structural control and robotics with high performance requirements. It has been a typical practice that dynamic models of mechanical structures are substantially simplified and modeling errors are compensated for by sophisticated control design. In practical uses, as we have experienced, feedback control elements will reflect overall system responses and undesired high frequency behavior may be induced if the dynamic model is over simplified.

Recently, efforts have been made to accurately model and control flexible structural systems by dealing with the distributed nature of them. Some researchers concentrated on formulating the distributed parameters analytically using direct formulation,[1,2] transfer function,[3] and Lyapunov's method,[4] etc. Others were attempting to deal with more complex structures using a numerical approach and developed approximate models by discretizing those mechanical parts into finite numbers of time-dependent parameters. This includes the lumped parameter approach,[5] Rayleigh-Ritz method,[6] assumed mode method,[7-9] and finite element method,[10-13] etc.

Most of the discretized parameter approaches solve the control dynamics in a two-step way by first trying to obtain a simplified but rational mechanical model and then coupling this model with controllers to simulate the entire system. In such an approach, the following situations may happen: the interactions between controllers and structural parts may be ignored and spill-over problems may occur,[14] and only a few degrees of freedom (DOFs) of the structure are concerned.[15] For these reasons, it is preferred to model the structure and control actuator simultaneously and completely. The finite element modeling,[12,13] state-space representation,[16] transfer matrix approach,[17] and their combinations[18-20] were presented for this purpose.

The finite element method (FEM) is probably the most widely used approach for structural analyses. Direct use of it may have restrictions in translating electric controllers into equivalent structural components.[12,13] A more satisfactory solution is to use FEM in state-space form so that the control parts can be included.[18] However, for larger systems, the application of FEM often leads to prohibitive time and cost of computation. As another discipline of numerical approach in the area of structural dynamics, the transfer matrix method (TMM) was well developed for the purpose of reducing the matrix size and saving computational efforts.[21] It allows the analysis of a system, chainlike or branching, to be affected by subsequent translation of characteristic section state vectors (forces and displacements) from the initial section of the system to the last one while the matrix size of the system remains the same. Book et al.[17] applied this approach to their controller design of a multilink flexible manipulator. In their study, the application of TMM was limited to collocated control with spring-type controllers and the control signals had no way to be input to the system. For the purpose of coupling both the advantages of FEM and TMM, a combined finite element transfer matrix method (FETMM) for structural analysis was presented.[22,23] This FETMM has the advantage of accurately modeling a dynamic system and retaining small matrix size. Our previous work[20] further extended this FETMM to have systematic capabilities of modeling a controlled single flexible link system in which the control DOFs are incorporated.

In this presentation, a systematic FETMM is first presented for the modal analysis of a controlled multilink system with both joint and structural flexibility. Both revolute and pris-

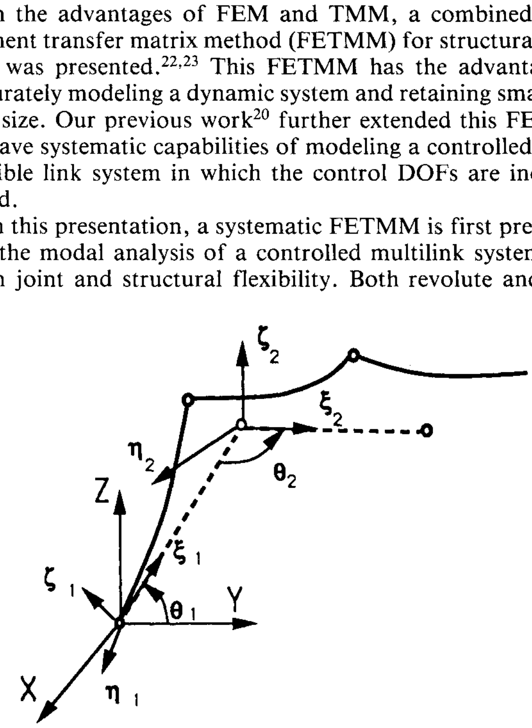

Fig. 1 Controlled multilink manipulator and the rigid manipulator-based reference coordinates.

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matic joints, with collocated or noncollocated control, can be simulated. An example of a two-link/two-joint control system with both joint and link flexibilities is presented to illustrate the availability of this proposed scheme. The effects of mechanical-control interactions are also illustrated.

II. Kinematics and Dynamics

Referring to Fig. 1, a controlled multilink manipulator with both joint and structural flexibilities is analyzed. For the purpose of modal analysis, the rigid manipulator-based reference coordinates shown in that figure are adopted for simplicity to represent link kinematics, and the nonlinear effects due to base motion and manipulator deflection are neglected. By this simplification, the following transformation between the proximate end of a current link and the distant end of the following link is adopted:

$$[R] = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & 0 \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\cos\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (1)$$

where α_i, not shown in the figure, is the twist angle between two joint axes.

Now, by discretizing a flexible arm into finite elements as shown in Fig. 2, and applying the well-known standard FEM procedures to formulate the differential segment dynamics, the element dynamic equation with respect to element local coordinate is obtained as

$$[m]_e\{\ddot{u}\}_e + [c]_e\{\dot{u}\}_e + [k]_e\{u\}_e = \{f\}_e + \{\hat{f}\}_e \qquad (2)$$

where

$[m]_e$ = element mass matrix
$[c]_e$ = element damping matrix
$[k]_e$ = element stiffness matrix
$\{u\}_e$ = element displacement vector referring to element coordinate system
$\{f\}_e$ = element force vector referring to element coordinate system
$\{\hat{f}\}_e$ = control actuating force vector referring to element coordinate system

The driving force vector $\{\hat{f}\}_e$ results from the combined activities of control elements, feedback elements, and the mechanical states that are fed back to control loops:

$$\{\hat{f}\}_e = [G_c]\{i_r\} + [M_{fb}]\{\ddot{u}\}_e + [C_{fb}]\{\dot{u}\}_e + [K_{fb}]\{u\}_e \qquad (3)$$

where

$[G_c]$ = forward transfer matrix of the control system
$[M_{fb}]$ = acceleration feedback transfer matrix of the control system
$[K_{fb}]$ = displacement feedback transfer matrix of the control system
$\{i_r\}$ = reference input signals to the controller

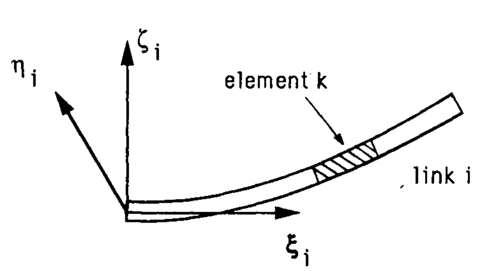

Fig. 2 Typical element on a manipulator link.

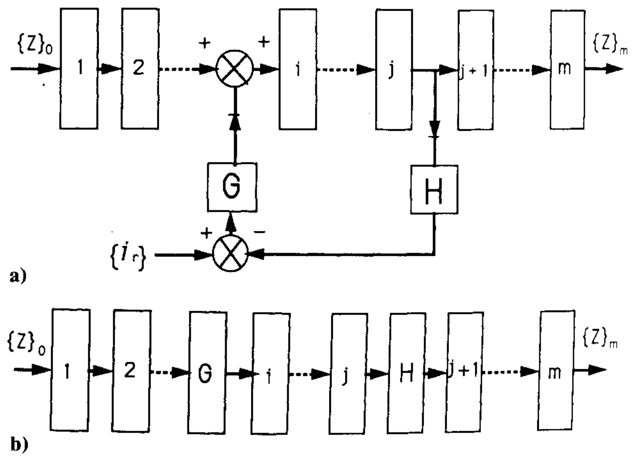

Fig. 3 Finite element transfer matrix modeling of a controlled flexible manipulator.

III. Finite Element Transfer Matrix Formulation

Transfer Matrix Techniques for a Multilink System

To model a controlled multilink system, some further strategies in addition to those of conventional TMM are applied as follows:

1) Finite element techniques are applied to effectively model the structural dynamics.

2) Similar to structural DOFs, control reference signals are treated as part of system states. Transfer matrices are modified to include control transfer functions by introducing the different coupling characteristics between control and mechanical parts.

3) Control actuator and feedback block are treated as an individual transfer element instead of merging them into a single spring-type control element. This causes the modeling of a noncollocated control to be possible.

4) Control actuating torques are treated as internal forces that have action and reaction components. Hence two control actuating elements, namely, action and reaction, are modeled simultaneously for a manipulator joint.

Following the conventions used in TMM, each link is divided into a certain number of subsegments connected by transfer states, as shown in Fig. 3. These segments can be the finite elements or substructures. In addition to structural elements, control blocks are also considered as transfer elements, and control signals together with mechanical DOFs are incorporated to form a transfer state vector. This state vector can be written in Laplace domain as

$$\{z\} = \left\{\{U\}', \{F\}', \{\Delta U\}', \{i\}'\right\}' \qquad (4)$$

In addition, the input-to-output relation of any transfer element can be represented as

$$\{z\}_i = [T]_i\{z\}_{i-1} \qquad (5)$$

or in partitioned form

$$\begin{Bmatrix} \{U\} \\ \{F\} \\ \{\Delta U\} \\ \{i\} \end{Bmatrix}_i = \begin{bmatrix} [T]_{11} & [T]_{12} & [T]_{13} & [T]_{14} \\ [T]_{21} & [T]_{22} & [T]_{23} & [T]_{24} \\ [T]_{31} & [T]_{32} & [T]_{33} & [T]_{34} \\ [T]_{41} & [T]_{42} & [T]_{43} & [T]_{44} \end{bmatrix}_i \begin{Bmatrix} \{U\} \\ \{F\} \\ \{\Delta U\} \\ \{i\} \end{Bmatrix}_{i-1} \qquad (6)$$

where $[T]_i$ is the overall transfer matrix of the ith segment; $\{z\}_{i-1}$ is the input state vector to the ith segment, $\{z\}_i$ is the output state vector from that segment; $\{U\}$, $\{F\}$, and $\{\Delta U\}$ are Laplace transforms of displacements states, force states, and joint discontinuities, respectively (both translational and rotational DOFs are included); $\{i\}$ are control signals, and $[T]_{ij}$ represents the transfer matrices of associated DOFs. Once the transfer matrices of all the system elements have been formulated, the overall transfer matrix equation can be obtained by passing along from the first section to the last one of that integrated system:

$$\{z\}_m = ([T]_m[T]_{m-1}\cdots[T]_2[T]_1)\{z\}_0 = [\hat{T}]\{z\}_0 \tag{7}$$

This equation relates the input state vectors to the outputs of the integrated system. The states consist of the mechanical boundary DOFs and the control signals at the boundary.

Before proceeding to formulate the associated transfer matrices of control and mechanical elements, some remarks have to be made. First, although the transfer matrix (7) has a similar form to that of the block transfer functions of control, Eq. (7) transfers power flows, but the control transfer functions only relate signals. Second, there are high input but low output impedances for control elements, whereas for structure elements the impedances are matching. Consequently, for control elements they are unilaterally coupled, but for structure elements the couplings are bilateral. Because of these differences, some modifications on transfer matrices of the conventional FETMM are necessary to model the structure and control in a systematic manner.

Transfer Matrix of a Structural Finite Element

For a structural element with no control forces acting, the equation of motion can be rewritten in Laplace domain as

$$(s^2\cdot[m]_e + s\cdot[c]_e + [k]_e)\{U\}_e = \{F\}_e \tag{8}$$

where $\{U\}_e$ and $\{F\}_e$ are the Laplace transforms of $\{u\}_e$ and $\{f\}_e$, respectively. Equation (8) can also be rewritten as

$$[D]_e\{U\}_e = \{F\}_e \tag{9}$$

or

$$\begin{bmatrix}[D]_{11} & [D]_{12}\\ [D]_{21} & [D]_{22}\end{bmatrix}_e \begin{Bmatrix}\{U\}_1\\ \{U\}_2\end{Bmatrix}_e = \begin{Bmatrix}\{F\}_1\\ \{F\}_2\end{Bmatrix}_e \tag{10}$$

where $[D]_e = (s^2\cdot[m]_e + s\cdot[c]_e + [k]_e)$ and is called the element dynamic matrix. Now, for a nonsingular $[D]_{12}$, by expanding Eq. (10) and solving for $\{U\}_2$ and $\{F\}_2$ in terms of $\{U\}_1$ and $\{F\}_1$, we may obtain the element transfer matrix equation as follows:

$$\begin{bmatrix}[T]_{11} & [T]_{12}\\ [T]_{21} & [T]_{22}\end{bmatrix}_e \begin{Bmatrix}\{U\}_1\\ \{F\}_1\end{Bmatrix}_e = \begin{Bmatrix}\{U\}_2\\ \{F\}_2\end{Bmatrix}_e \tag{11}$$

where

$$[T]_{11} = -[D]_{12}^{-1}\cdot[D]_{11}, \qquad [T]_{12} = [D]_{12}^{-1}$$
$$[T]_{21} = [D]_{22}\cdot[D]_{12}^{-1}\cdot[D]_{11} - [T]_{21}, \quad [T]_{22} = -[D]_{22}\cdot[D]_{12}^{-1} \tag{12}$$

If $[D]_{12}$ is singular, Eq. (11) may be obtained by direct formulations.

The preceding formulations are developed for structural DOFs only. For an integrated control system, however, since there is not any influence on the control signals in a pure structural element with control-mechanical interactions, the control signals may be treated as "through" states in such an element, and hence the transfer matrix equation can simply be obtained from Eq. (11) as

$$\begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_i = \begin{bmatrix}[T]_{11} & [T]_{12} & [0] & [0]\\ [T]_{21} & [T]_{22} & [0] & [0]\\ [0] & [0] & [I] & [0]\\ [0] & [0] & [0] & [I]\end{bmatrix}_i \begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_{i-1} \tag{13}$$

Transfer Matrices of Control Transfer Elements

There are two kinds of control-mechanical interface elements: the control actuating elements and the control feedback elements. For a control actuating interface, since the actuating forces are summed to the original force state of the structure and no reaction forces are applied back to the signals of the control loop, this kind of junction may be replaced by a simple transfer element $[T_G]$ as shown in Fig. 4a. The transfer matrix equation is given as

$$\begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_i = \begin{bmatrix}[I] & [0] & [0] & [G_1]\\ [0] & [I] & [0] & [G_2]\\ [0] & [0] & [I] & [G_3]\\ [0] & [0] & [0] & [G_4]\end{bmatrix}_i \begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_{i-1} \tag{14}$$

where $[G_1],\ldots,[G_4]$ are the associated control transfer functions.

Similarly, for the feedback interface, to relate the control signals $\{i\}^*$ to the reference inputs $\{i_r\}$, we have the following equation:

$$\begin{aligned}\{i\}^* &= \{i_r\} - \{i\}_{fb}\\ &= \{i_r\} - [H_1]\{U\} - [H_2]\{F\} - [H_3]\{\Delta U\}\end{aligned} \tag{15}$$

The feedback junction can be replaced by an alternative transfer element as shown in Fig. 4b, and the transfer matrix equation becomes

$$\begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_i = \begin{bmatrix}[I] & [0] & [0] & [0]\\ [0] & [I] & [0] & [0]\\ [0] & [0] & [I] & [0]\\ [H_1] & [H_2] & [H_3] & [H_4]\end{bmatrix}_i \begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_{i-1} \tag{16}$$

where $[H_1],\ldots,[H_4]$ are the associated control feedback transfer functions and $\{i\}_i = \{i_r\}$, $\{i\}_{i-1} = \{i\}^*$, $[H_4] = [1]$ for this case.

For those control blocks with no control-mechanical interactions, since only control signals are operated, their transfer

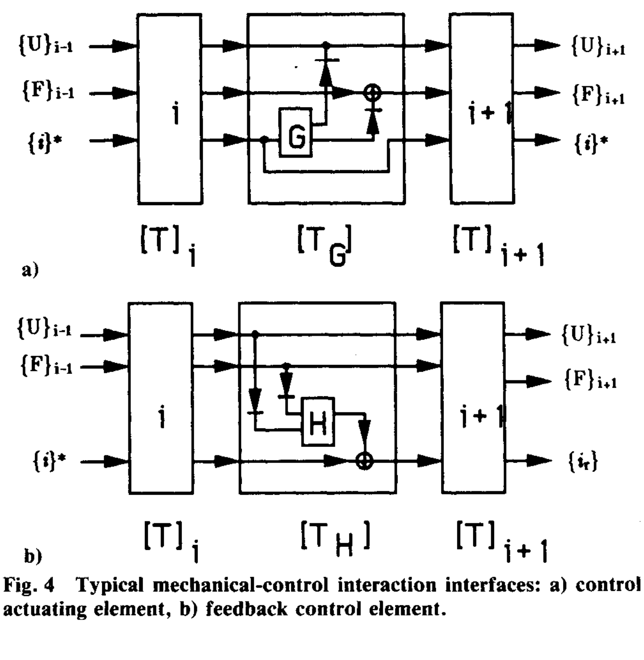

Fig. 4 Typical mechanical-control interaction interfaces: a) control actuating element, b) feedback control element.

matrices can be formulated and simplified by conventional block techniques of classical control engineering.

Transfer Matrices of Joint Elements

Because there are some displacement discontinuities (for prismatic joint) or rotational discontinuities (for revolute joint) at a joint, the associated matrix equation can be written as

$$\begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_i = \begin{bmatrix}[I]&[0]&[0]&\vdots&[0]\\ [0]&[I]&[\delta]&\vdots&[0]\\ [0]&[0]&[I]&\vdots&[0]\\ [0]&[0]&[0]&\vdots&[I]\end{bmatrix}_i \begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_{i-1} \qquad (17)$$

where $[\delta]$ is the discontinuity operator. The associated coordinate transfer matrix at the joint, which translates the system states from the proximate end of a current link to the distant end of the following link under the constraints of equilibrium and compatibility, may be written as

$$\begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_i = \begin{bmatrix}\multicolumn{2}{c}{\multirow{2}{*}{$[\;R\;]$}}&[0]&\vdots&[0]\\ &&[0]&\vdots&[0]\\ [0]&[0]&[I]&\vdots&[0]\\ [0]&[0]&[0]&\vdots&[I]\end{bmatrix}_i \begin{Bmatrix}\{U\}\\ \{F\}\\ \{\Delta U\}\\ \{i\}\end{Bmatrix}_{i-1} \qquad (18)$$

where $[R]$ is a coordinate transform matrix referring to the rigid manipulator references.

System Formulation

Referring to Fig. 3, the following procedures, based on Eqs. (13–18), may be applied for modal analysis of a controlled multilink system with both joint and structural flexibilities:

1) Discretize the structural system into transfer elements as shown in Fig. 3a, and formulate the corresponding transfer matrices as in Eq. (13).

2) For flexible joints, the joint flexibility is represented by an equivalent structural transfer element.

3) Formulate and simplify the transfer functions of the control system to its compact form as shown in Fig. 3a.

4) Using Eqs. (14) and (16) and replacing the control-mechanical junctions by their equivalent transfer elements, as shown in Fig. 3b, a chainlike system is obtained.

5) Passing along from the first segment to the last one, the transfer matrix equation of the integrated system is obtained as

$$\{z\}_m = ([T]_m \cdot [T]_{m-1} \cdots [T_H] \cdots [T_G] \cdots [T]_2 \cdot [T]_1)\{z\}_0$$
$$= [\hat{T}]_m \cdot \{z\}_0 \qquad (19)$$

6) Substitute all of the mechanical boundary conditions.

7) Setting the reference control inputs $\{i_r\}$ equal to $\{0\}$, Eq. (19) becomes an eigenvalue problem. The eigenvalues (poles) and the associated eigenvectors (mechanical mode shapes) can be easily obtained by the same way as the conventional TMM does.

8) To obtain the transfer function of the integrated system, set $\{i_r\}$ equal to unity $\{1\}$. The transfer function matrices relating outputs to inputs can be figured out easily. After some matrix operations ($\{z\}_1 = [T]_1\{z\}_0$, $\{z\}_2 = [T]_2\{z\}_1$, ..., etc.), transfer functions of the integrated system are obtained completely.

9) To calculate the system response corresponding to arbitrary control inputs $\{i_r\}$, just introduce this input vector into Eq. (19) and calculate the responses in the same way as in step 8.

It is noted that in Eq. (19), all of the system parameters, both the mechanical and control, are included simultaneously, and all of the mechanical state responses can be found completely and systematically. Moreover, the final matrix size to solve the system depends on the DOFs of each segment state regardless of how many segments the system is discretized. Thus it has the advantage of simulating more complex mechanical-control systems and still keeping a small matrix size. In addition, this method is not confined to solving the collocated control problems. It can also be applied to simulate a noncollocated control system with flexible structure components as shown in the following illustrative problem.

Note also that this proposed FETMM scheme can easily be reduced to the model of Book et al.[17] for the special case of collocated spring-type controllers. It can be done by omitting the control and discontinuity DOFs and then applying the control effects on the associated mechanical DOFs.

Table 1 Amplitudes, system parameters

Link length	$l_i = 8.13$ m
Link density	$\rho_i = 4.8$ kg/m
Link stiffness	$EI_i = 7.99 \times 10^5$ N $\cdot$ m^2
Joint masses	$m_1 = 53.1$ kg, $m^2 = 226.0$ kg, $J_0 = 1240.0$ kg $\cdot$ m^2
	$J_{11} = 4.57$ kg $\cdot$ m^2, $J_{12} = 585.0$ kg $\cdot$ m^2
	$J_2 = 68.3$ kg $\cdot$ m^2

IV. Illustrative Application

As an illustrative example, a planer two-link/two-joint manipulator system, as shown in Fig. 5a, is analyzed. The manipulator is assumed to have Euler-Bernoulli links and have controllers with proportional differential (PD) gain and unit feedback at the joints. The associated system parameters are listed in Table 1. For simplicity, the lumped parameter model shown in Fig. 5b is adopted to describe the elastic manipulator. The external effects due to gravity and rigid motion are neglected. Since there are two joints, the state vector of this system (4) is written as

$$\{z\} = \{U_x, U_y, \theta, F_x, F_y, \tau, \Delta\theta, i_1, i_2\}^t \qquad (20)$$

and the associated transfer matrices in Eqs. (13–16) are given as follows:

Lumped masses[21]:

$$\begin{bmatrix}[T]_{11} & [T]_{12}\\ [T]_{21} & [T]_{22}\end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ -ms^2&0&0&1&0&0\\ 0&-ms^2&0&0&1&0\\ 0&0&-Js^2&0&0&1\end{bmatrix} \qquad (21)$$

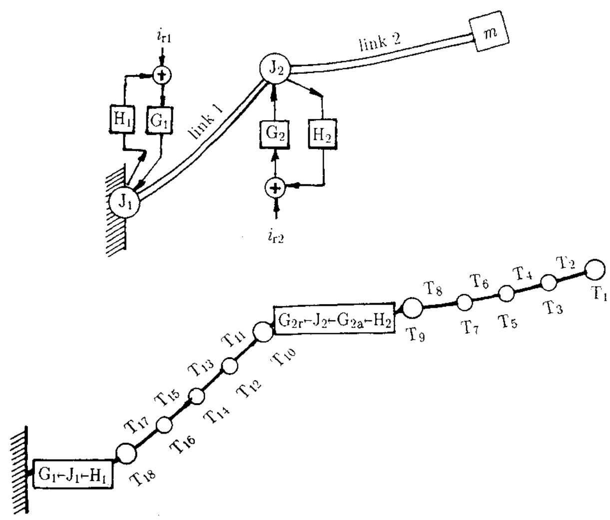

Fig. 5 Case application: a controlled two-link/two-joint flexible manipulator.

Elastic links[21]:

$$\begin{bmatrix} [T]_{11} & [T]_{12} \\ [T]_{21} & [T]_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -l & 0 & Bl^3 & 3Bl^2 \\ 0 & 0 & 1 & 0 & -3Bl^2 & -6Bl \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & l & 1 \end{bmatrix} \tag{22}$$

Control elements:

$$[G_1] = [G_3] = [H_2] = [0]$$

$$[G_4] = [H_4] = [1]$$

$$[G_1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ K_{v1}s + K_{f1} & 0 \end{bmatrix} \text{ for joint 1}$$

$$[G_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & K_{v2}s + K_{f2} \end{bmatrix} \text{ for joint 2}$$

$$[H_1] = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}, \quad [H_3] = [0] \quad \text{for joint 1}$$

$$[H_1] = [0],\ [H_3] = \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix} \quad \text{for joint 2} \tag{23}$$

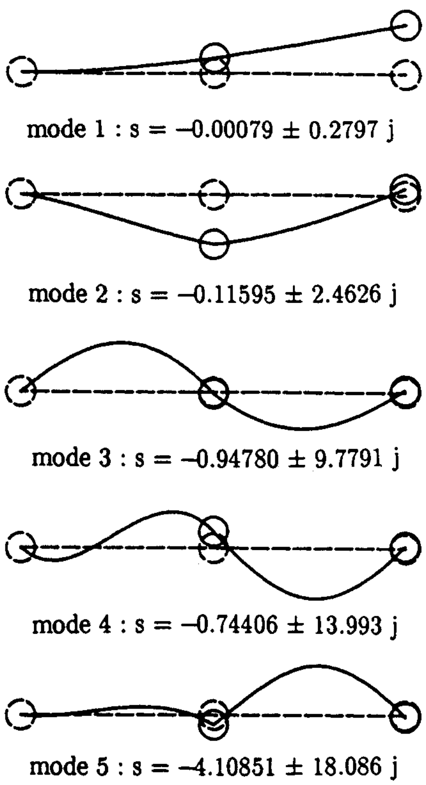

Fig. 6 Mode shapes of the controlled two-link/two-joint system.

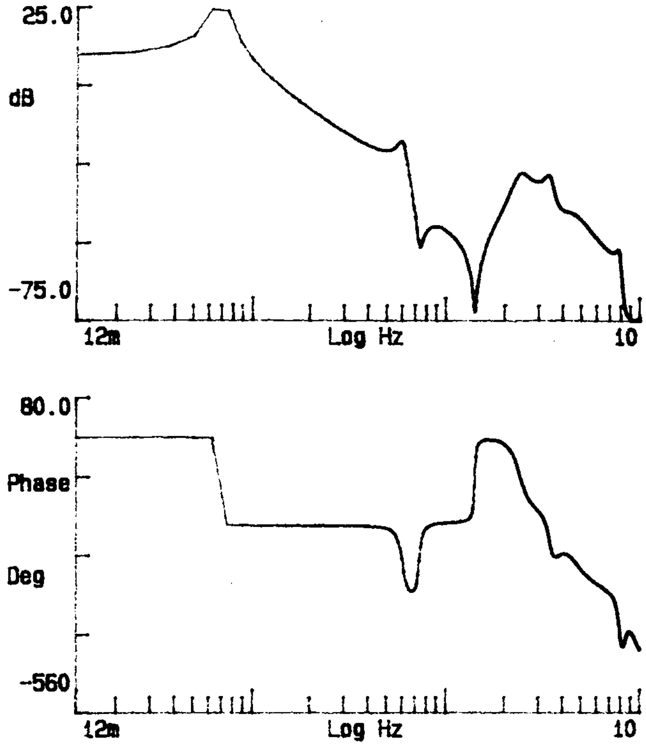

Fig. 7 Obtained frequency response of the controlled manipulator at tip position.

where $B = 1/6EI$, $H = -1$, and K_v and K_f are control gains. Finally, a 9×9 transfer matrix equation of the system is obtained as

$$\{z\}_{\text{base}} = ([T_{H_1}][T_{G_1}][T_{J_1}][T_{18}]\ldots[T_{10}][T_{H_2}] \times [T_{G_{2r}}][T_{J_2}][T_{G_{2a}}][T_9]\ldots[T_1])\{z\}_{\text{tip}} \tag{24}$$

A general program in C language with double precision floating points has been developed to derive and solve the polynomial transfer matrices. All of the solutions are solved in polynomial form first and then the eigenvalue problem and response simulation are solved by applying Newton's method and other conventional numerical techniques. Finally the time domain solutions are obtained directly by the inversion of the fast Fourier transform.

Parameter Normalization and Boundary Conditions Implementation

For improving numerical conditioning, all of the system parameters are normalized first. The following physical quantities are applied for this normalization:

Stiffness: $(EI)_0$, FL^2; length l_0; mass per unit length μ_0, ML^{-1}; and time ω_0, s^{-1} where

$(EI)_0 = E_1 I_1$,
$l_0 = l_1 + l_2$
$\mu_0 = (\mu_1 l_1 + \mu_2 l_2)/l_0$
$\omega_0 = (E_1 I_1/\mu_0 l_0^4)^{1/2}$
E = Young's modulus
I = area moment of inertia of the cross section

To implement the boundary conditions, the following are noted:

1) There is one additional boundary condition at the joint location, i.e., $\tau = 0$ if the actuator is in action, or $\Delta\theta = 0$ if the actuator is braked.

2) It is necessary to implement those known input signals as part of the boundary conditions.

After substituting the appropriate boundary coniditions into Eq. (24) and rearranging it, a 4×4 matrix equation for solving the problem is obtained. This matrix size is relatively small and easy to deal with.

Manipulator Mode Shapes of the Integrated System

Figure 6 shows the eigensolutions of this manipulator system, in which the control parts are coupled. Instead of giving zero applied torques to the joints, since the real inputs to the manipulator system are those control reference signals, these eigensolutions are obtained by setting $i_{r1} = i_{r2} = 0$. From these solutions, it is seen that the control parts serve as intermediate springs at the joints between two adjacent links. This is different from using free or fixed joint conditions as usually adopted in manipulator modal analysis. Note that Fig. 6 gives only the first five eigenmodes of the integrated system. For higher modes, numerical ill-conditioning occurs. This ill-conditioning results from omitting the higher order terms with a relatively small coefficient in polynomials in formulating the system transfer equations. However, it is fortunate that the effects of ill-conditioning on lower modes are usually not significant, while these modes are frequently the most interesting ones in controller design. Besides, techniques of order reduction for transfer function[24,25] may also be applied to diminish this ill-conditioning.

The frequency responses of all of the system states can be obtained simultaneously. Figure 7 shows one of them (at the tip position) with $i_{r1} = i_{r2} = 1$. For other combinations of i_{r1} and i_{r2} (for example, $i_{r1} = 0$ and $i_{r2} = 1$, etc.), the associated frequency responses can also be obtained easily without reformulating the system equations.

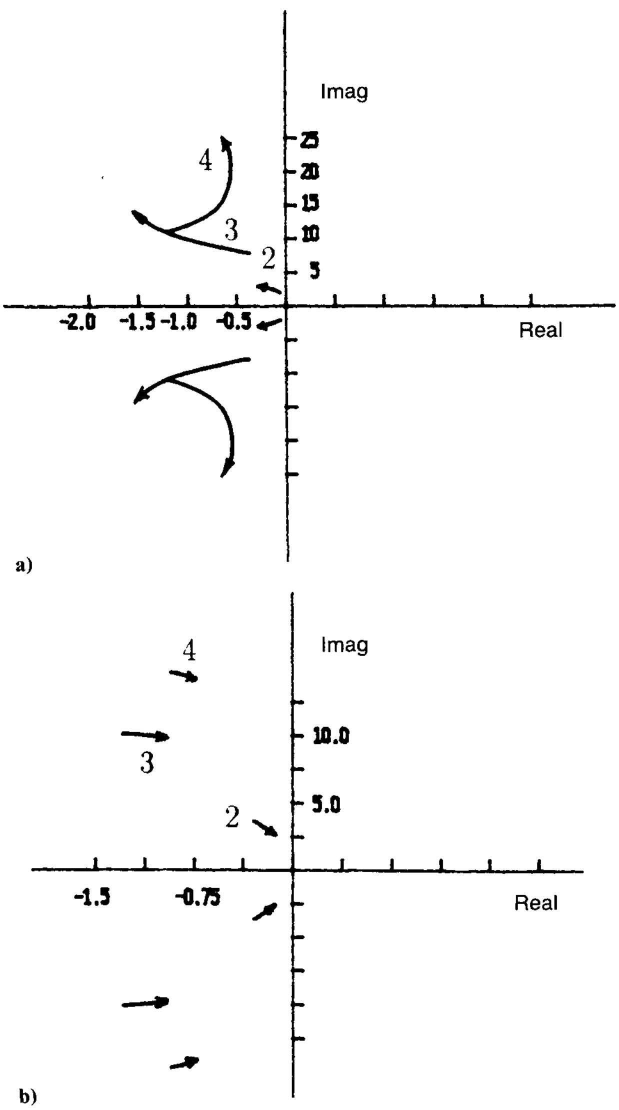

Fig. 8 Root locus for varying mechanical properties: a) link stiffness EI = **1.0 ~ 5.0, b) tip loading** m **= 0.0 ~ 2.0**

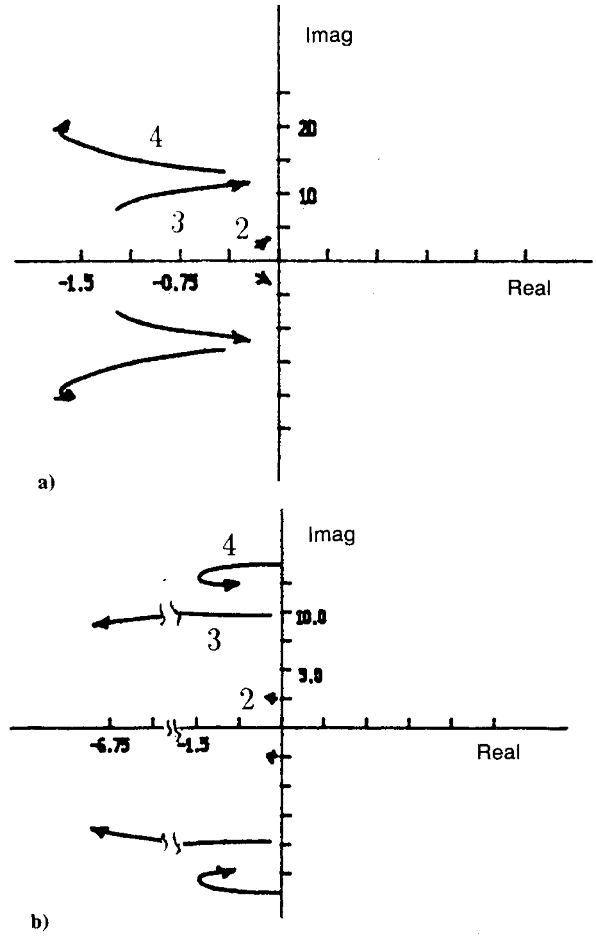

Fig. 9 Root locus for increasing control gains at joint 2: a) K_{f2} = **1.0 ~ 21.0, b)** K_{v2} = **0.0 ~ 0.1.**

Parameter Investigation of the Control Actuator and Flexible Manipulator

The proposed FETMM also has merit for systematic investigations of integrated control and flexible link systems. Figure 8 sketches the root loci of the multilink system by changing mechanical properties (link stiffness and tip loading). Figure 9 also sketches the root loci with various control gains at joint 2. These figures give the sensitivities of modes 2, 3, and 4. Mode 1 is so slightly changed relative to other modes and cannot be seen in the figures. From these figures, it is seen that each system parameter has its tendencies of sensitivity on eigenmodes. Based on these sensitivities, one can obtain a satisfactory and optimized multilink control system accounting for both structural and joint flexibilities.

Figure 10 shows how the frequency responses of the system will be affected when the system parameter K_{v1} is varied. Figure 11 also shows how the mode shapes of the system may be affected by changing the system parameters. The results are obtained by varying the control gain of the base joint (K_{f2} of joint 2), in which the mode shape of link 1 is significantly changed. The results in Figs. 10 and 11 show that the mechanical eigenmodes may be changed by applying different control parameters. The reason is that the control parts in this system play the roles of joint springs and dampers. As we change the control parameters, the stiffness and damping of the system are also changed. This is why it is preferred to simulate the integrated system in a systematic way, in which all of the

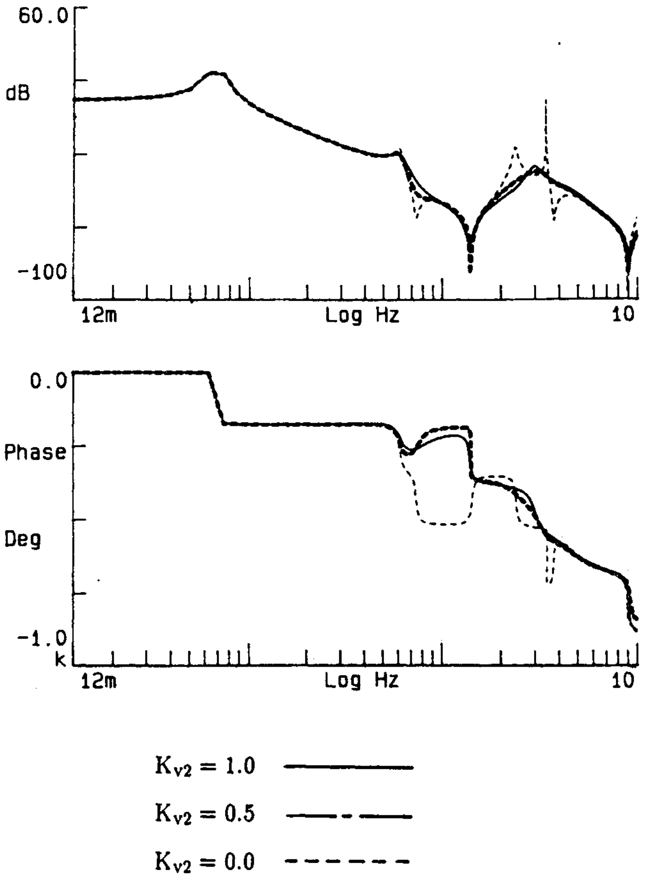

Fig. 10 Example of effects on frequency response function (at link tip): affected by control damping.

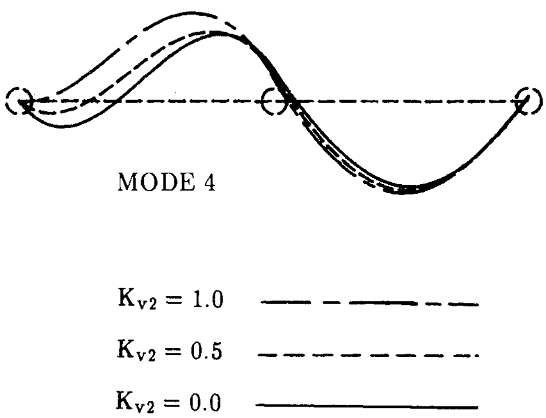

Fig. 11 Example of effects on mode shapes: affected by control damping.

system parameters can be analyzed altogether. Otherwise problems of spillover may occur.

Elastic Joint Effects and Noncollocated Control

In addition to structural flexibility, this FETMM approach can easily be applied to simulate a multilink control system with join flexibility. It also has the capability of simulating the noncollocated control systems that cannot be done by the TMM approach of Book et al.[17]

Suppose that the base joint (joint 1) is flexible with stiffness 0.1 EI_0. The resulting impulse response time history of the link tip is sketched in Fig. 12a. In this case, the feedback sensor at that joint is collocated with the actuator. We see that the coupling of joint flexibility causes a long period oscillation of

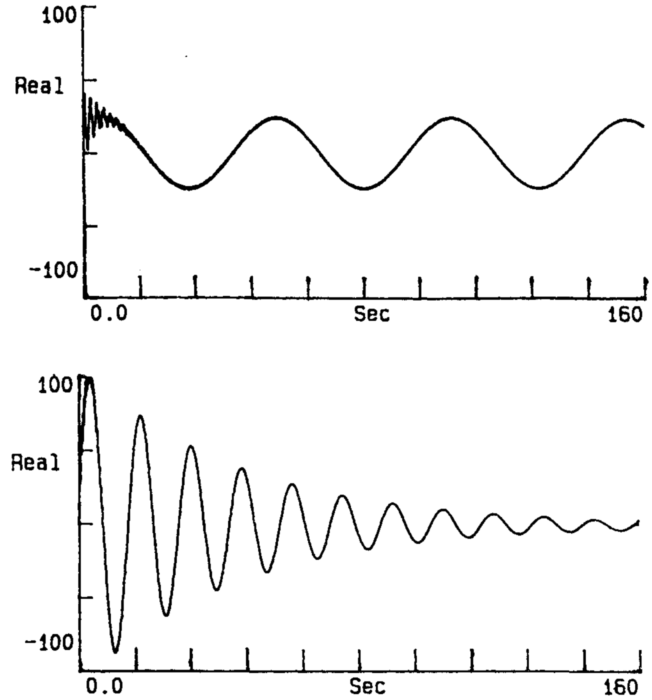

Fig. 12 Manipulator with flexible base joint: impulse response at the tip of link 2; a) collocated control, b) noncollocated control.

the link tip. Now, suppose that we have the feedback sensor placed in front of the joint so that the actuator and feedback sensor are noncollocated. Then the situation of tip oscillation is improved as shown in Fig. 12b. These results indicate that the associated effect of this noncollocated control is to make the system stiffer.

V. Conclusions

A systematic finite element transfer matrix approach is presented for the modal analysis of an integrated multilink control system with both joint and structural flexibility. Since it is systematic, the effects of interaction between structures and controllers can be considered. In this approach, the system state vectors incorporate both the control signals and mechanical degree of freedoms. Hence the control reference signals can be directly input to the system without calculating equivalent static actuating torques. The simulation of noncollocated control is also available.

An example of a two-link, two-joint control system is given to illustrate the availability of this finite element transfer matrix approach. In this example, some parametric sensitivity analyses of both structural and control design parameters in a systematic manner are presented. The results indicate that control parameters may affect mechanical eigenmodes. This illustrated example also shows its capability in the simulation of noncollocated control in the case when joint flexibility is involved. The results indicate that such a noncollocated control may stiffen the link system. Furthermore, since the transfer matrix method is itself an exact dynamic condensation technique to reduce the degres of freedom of a dynamic model, this proposed finite element transfer matrix approach also provides the merits of remarkably reducing the matrix size of system equation of the integrated control system regardless of how many transfer elements are discretized.

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